Field Theory of Quantum Hall Effects, Composite Bosons, Vortices and Skyrmions

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Abstract

A field theory of quantum Hall effects is constructed based on the composite-boson picture. It is tightly related with the microscopic wave-function theory. The characteristic feature is that the field operator describes solely the physical degrees of freedom representing the deviation from the Laughlin state. It presents a powerful tool to analyze excited states within the lowest Landau level. It is shown that all excitations are nonlocal topological solitons in the spinless quantum Hall system. On the other hand, in the presence of the spin degree of freedom it is shown that a quantum coherence develops spontaneously, where excitations include a Goldstone mode besides nonlocal topological solitons. Solitons are vortices and Skyrmions carrying the U(1) and SU(2) topological charges, respectively. Their classical configurations are derived from their microscopic wave functions. The Skyrmion appears merely as a low-energy excitation within the lowest Landau level and not as a solution of the effective nonlinear sigma model. We use it as a consistency check of the Skyrmion theory that the Skyrmion is reduced to the vortex in the vanishing limit of the Skyrmion size. We evaluate the activation energy of one Skyrmion and compare it with experimental data.

1 Introduction

The quantum Hall (QH) effect [1] is a remarkable macroscopic quantum phenomenon observed in the two-dimensional electron system at low temperature T and in strong magnetic field B. The Hall conductivity is quantized with extreme accuracy and develops a series of plateaux at magic values of the filling factor $\nu = 2\pi\hbar\rho_0/eB$. Here, ρ_0 is the average electron density. The longitudinal resistivity is

$$\rho_{xx} \propto \exp\Big(-\frac{\Delta}{2k_BT}\Big), \tag{1.1}$$

with k_B the Boltzman constant and Δ the activation energy of a quasihole-quasielectron pair, $\Delta = \Delta_h + \Delta_e$.

It is widely recognized that the QH effect comes from the realization of an incompressible ground state. The composite-boson picture has proved to be quite useful to understand all essential aspects of QH effects [2, 3, 4, 5]. Electrons may condense into an incompressible quantum Hall liquid as composite bosons. The QH state is such a condensate of composite bosons, where quasiparticles are vortices [6]. When the SU(2) symmetry is incorpolated, a quantum coherence may develop spontaneously [7, 8, 9], turning the QH system into a QH ferromagnet. New excitations are Skyrmions [10, 11]. They were initially introduced based on the composite-boson picture [10] and later studied also in a microscopic Hartree-Fock approximation [12]. However,

the composite-boson theory is not satisfactory at all. The naive formulation suffers from a serious problem we mention later. Furthermore, the relation between the composite-boson theory and the microscopic wave-function theory is unclear. For instance, although the Skyrmion wave function is reduced to the vortex wave function in the vanishing limit of the Skyrmion size [9], this property is lost for the Skyrmion and vortex classical fields in the effective nonlinear sigma (NLσ) model of Skyrmions [10, 9]. The aim of this paper is to present an improved scheme of composite bosons free from these difficulties.

The idea of the improved composite-boson theory is summarized as follows. For the sake of simplicity we consider the QH system neglecting the spin degree of freedom. We use the complex coordinate normalized as $z = (x + iy)/2\ell_B$ with ℓ_B the magnetic length. We are concerned about physics taking place within the lowest Landau level. Indeed, at sufficiently low temperature the relevant excitations are those confined within the lowest Landau level. At $\nu = 1/m$ (m odd), any state $|\mathfrak{S}\rangle$ in the lowest Landau level is described by a wave function,

$$\mathfrak{S}[\boldsymbol{x}] \equiv \langle 0|\psi(\boldsymbol{x}_1)\cdots\psi(\boldsymbol{x}_N)|\mathfrak{S}\rangle = \omega[z]\mathfrak{S}_{\rm LN}[\boldsymbol{x}], \tag{1.2}$$

with $\mathfrak{S}_{\rm LN}[\boldsymbol{x}]$ the Laughlin function [6],

$$\mathfrak{S}_{\rm LN}[\boldsymbol{x}] = \prod_{r<s}(z_r - z_s)^m e^{-\sum_{r=1}^{N}|z_r|^2}, \tag{1.3}$$

where $\omega[z] \equiv \omega(z_1, z_2, \cdots, z_N)$ is an analytic function symmetric in all N variables. There is every reason [1] to believe that the ground state is the Laughlin state with $\mathfrak{S}[\boldsymbol{x}] = \mathfrak{S}_{\rm LN}[\boldsymbol{x}]$. Since $\omega[z]$ is symmetric in N variables, it must be a wave function of certain bosons, to which we refer as *dressed composite bosons*. Namely, we propose an improved bosonization by a mapping from the fermionic wave function $\mathfrak{S}[\boldsymbol{x}]$ to a bosonic wave function $\mathfrak{S}_\varphi[\boldsymbol{x}]$,

$$\mathfrak{S}[\boldsymbol{x}] \mapsto \mathfrak{S}_\varphi[\boldsymbol{x}] \equiv \omega[z]. \tag{1.4}$$

Denoting its field operator by $\varphi(\boldsymbol{x})$ we expect

$$\mathfrak{S}_\varphi[\boldsymbol{x}] \equiv \langle 0|\varphi(\boldsymbol{x}_1)\cdots\varphi(\boldsymbol{x}_N)|\mathfrak{S}\rangle = \omega[z], \tag{1.5}$$

which we establish in Section 3. The ground state is extremely simple in terms of composite bosons, where $\mathfrak{S}_\varphi[\boldsymbol{x}] = 1$. The power of z_r in $\omega[z]$ is the angular momentum carried by the rth composite boson. Hence it is interpreted that the Laughlin state is the one where all composite bosons condense into the angular-momentum zero state. Similarly, the vortex state with $\omega[z] = \prod_r z_r$ is the one where all composite bosons condense into the angular-momentum one state.

It is a characteristic feature of this theory that composite bosons represent the physical degree of freedom describing solely the deviation $\omega[z]$ from the ground state. When the N-body wave function $\mathfrak{S}_\varphi[\boldsymbol{x}]$ is factorizable, $\omega[z] = \prod_r \omega(z_r)$, it follows from (1.5) that the one-point function is analytic,

$$\langle\varphi(\boldsymbol{x})\rangle = \omega(z). \tag{1.6}$$

This is a highly nontrivial requirement. It connects directly the classical field $\langle\varphi(\boldsymbol{x})\rangle$ of an excitation to its microscopic wave function $\omega(z)$. In particular, it determines how the electron density modulates around the zeros of $\omega(z)$. We demonstrate that all excitations are topological solitons (vortices) in the QH state when spins are neglected, and that there appears a Goldstone mode and topological solitons (vortices and Skyrmions) in the QH state when spins are taken into account.

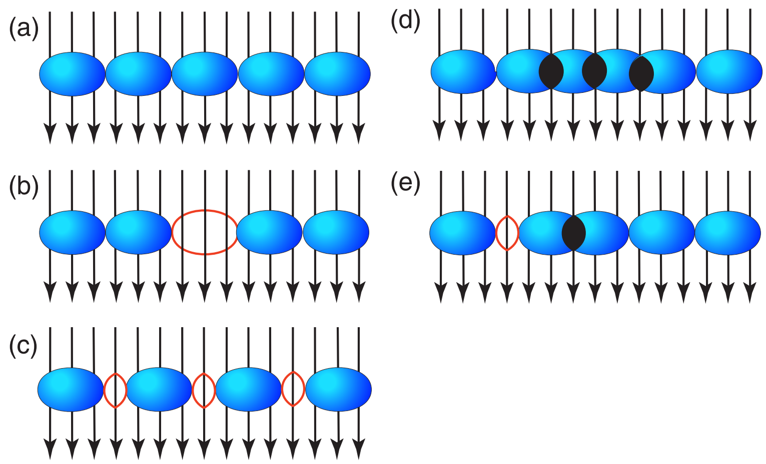

Figure 1: We illustrate the fractional QH state at $\nu = 1/3$. There are 3 flux quanta per electron. (a) The ground state is a closed packet of electrons pierced by 3 flux quanta. (b) When one electron is removed one big hole is created. (c) To lower the Coulomb energy there appear 3 small holes, which act as quasiholes. (d) When one electron is added there appear 3 overlapping of electrons, which act as quasielectrons. (e) Thirmal fluctuations create quasihole-quasielectron pairs.

Contrary to the common belief [10], the Skyrmion excitation appears merely as a low-energy excitation confined within the lowest Landau level and not as a solution of the effective NLσ model. Because of this origin the Skyrmion is naturally reduced to the vortex in the vanishing limit of the Skyrmion size.

The activation energy of the state $|\mathfrak{S}\rangle$ is given by the matrix element, $\langle H\rangle = \langle\mathfrak{S}|H|\mathfrak{S}\rangle$, with H the Hamiltonian. The calculation is quite intriguing due to the condition that the excitation is confined within the lowest Landau level. We have two complementary methods.

One method is to use a semiclassical approximation, which is appropriate to obtain the activation energy of a topological soliton. As we have stated, an investigation of the one-point function (1.6) reveals how the densiy modulation (and also the spin modulation in the QH ferromagnet) is induced around the topological soliton when the excitation is confined within the lowest Landau level. The excitation energy is calculable with the knowledge of the density modulation (and the spin modulation).

The other is an algebraic method based on the lowest-Landau-level (LLL) projection [13]. The electron coordinate $\boldsymbol{x} = (x, y)$ is decomposed into the guiding center (center-of-mass coordinate) $\boldsymbol{X} = (X, Y)$ and the relative coordinate $\boldsymbol{R} = (R_x, R_y)$. Because the physics in the lowest Landau level involves only the guiding center $\boldsymbol{X}$, the symmetry of the two-dimensional space is subject to the magnetic translation group, which is generated by the magnetic translation $e^{i\boldsymbol{qX}}$ and not by the Abelian translation $e^{i\boldsymbol{qx}}$. It has a crucial consequence [8, 9] that the spin density SU(2) operator and the electron density U(1) operator become noncommutative, because X and Y coordinates of the guiding center $\boldsymbol{X}$ are noncommutative. It implies that a spin rotation induces an electron density modulation and hence requires a Coulomb exchange energy, as is consistent with the semiclassical result. Making a perturbative expansion of the spin texture around the ground state, we derive the NLσ model from the matrix element $\langle\mathfrak{S}|H|\mathfrak{S}\rangle$ as the effective Hamiltonian describing perturbative spin fluctuations in QH ferromagnets [8, 9].

Before we present a detailed discussion of a field theory of composite bosons, let us describe a

physical picture of the fractional QH system. We neglect the spin degree of freedom for simplicity. All electrons are assumed to be in the lowest Landau level. Due to the Pauli exclusion principle only one electron can occupy one quantum-mechanical state. The density of states in each energy level is $D_n = 1/(2\pi\ell_B^2)$, which is equal to the number of Dirac flux quanta passing through unit area, $D_n = B/\phi_D$, where the Dirac flux quantum is $\phi_D \equiv 2\pi\hbar/e$. The filling factor of the energy level is thus defined by $\nu = \rho_0/D_n = 2\pi\hbar\rho_0/eB$. It is characteristic that there are m flux quanta per electron, $B/\rho_0 = m\phi_D$, at $\nu = 1/m$. We are able to attach m flux quanta to one electron by way of a phase transformation [14], composing a composite particle. It is a composite fermion [15] for even m, while it is a composite boson for odd m. Composite particles feel the effective magnetic field, which vanishes at $\nu = 1/m$. When m is odd, composite bosons become free and undergo a bose condenstation at $\nu = 1/m$. We may view that the condensate is a closed packet of composite bosons pierced by m flux quanta, as is illustrated in Fig.1(a). It is the nondegenerate ground state with homogeneous electron density. When one electron is removed, one "big" hole would appear in the homogeneous electron density as in Fig.1(b). It represent a charge defect e occupying three flux quanta. It is energitically favorable that it is dissociated into m "small" holes, each of which represents charge defect e/m pierced by one flux quantum, as in Fig.1(c). The small hole is not smeared out since it is combined with a flux quantum. Indeed, it is the vortex soliton (quasihole) carrying electric charge e/m, as we argue in the body. When one electron is added, there appears m overlapping of flux-electron composites as in Fig.1(d). Each of which represents charge excess $-e/m$ occupying one flux quantum. It is the antivortex soliton (quasielectron) carrying electric charge $-e/m$. Thirmal fluctuations generate quasihole-quasielectron pairs, as is illustrated in Fig.1(e).

One quasielectron (quasihole) carries the electric charge $-e/m$ $(+e/m)$ and the magnetic flux $-\phi_D$ $(+\phi_D)$ within the domain of size ℓ_B. Hence, the creation energy Δ_e (Δ_h) of a quasielectron (quasihole) is $\sim e^2/(m^2\varepsilon\ell_B)$. The system is said to be incompressible when there is a gap in the chemical potential as a function of the electron density. The fractional QH system is incompressible because the gap of the chemical potential is $m(\Delta_e + \Delta_h)$ at $\nu = 1/m$.

We next gives a physical picture of the QH ferromagnet at $\nu = 1$. When the Zeeman effect is small, each Landau level contains two almost degenerate levels with spin-up and spin-down states. All electrons are in the spin-up state. There is one flux quantum per electron, and the ground state is a closed packet of composite bosons pierced by one flux quantum. It is the nondegenerate ground state with homogeneous electron density. (If the Zeeman effect is absent, it is one of degenerate ground states.) When one electron is removed, one hole appears as in Fig.1(a). It is a vortex just as in the spinless QH state in Fig.1. The Coulomb energy is $\sim e^2/\varepsilon\ell_B$. When one electron is added, it is placed in the spin-down state as in Fig.1(a). It is a localized lump of a quantized electric charge, and its Coulomb energy is also $\sim e^2/\varepsilon\ell_B$. This is the picture of the integer QH state when the Zeeman energy is very large. When the Zeeman energy is negligible, however, it is possible to lower the Coulomb energy by rotating the spins of neighboring electrons, as in Fig.1(b). When the spin rotates, the electron density is also modulated coherently because the SU(2) and U(1) operators are noncommutative due to the magnetic-translation-group effect. As a result the electric charge is smeared into a wider domain. This is made possible by developing quantum coherence spontaneously. Resulting coherent excitations are Skyrmions. When the size of the Skyrmion is $\kappa\ell_B$, the Coulomb energy is deduced by factor $1/\kappa$. The size is determined by the competition between the Coulomb energy and the Zeeman energy. It is clear in this picture that the Skyrmion [Fig.1(b)] is reduced to the vortex [Fig.1(a)] when the Zeeman energy is sufficiently large.

This paper is composed as follows. In Section 2 composite-boson fields are defined. First, as in the standard composite-boson theory [3, 5], we attach an odd number of flux quanta to an electron

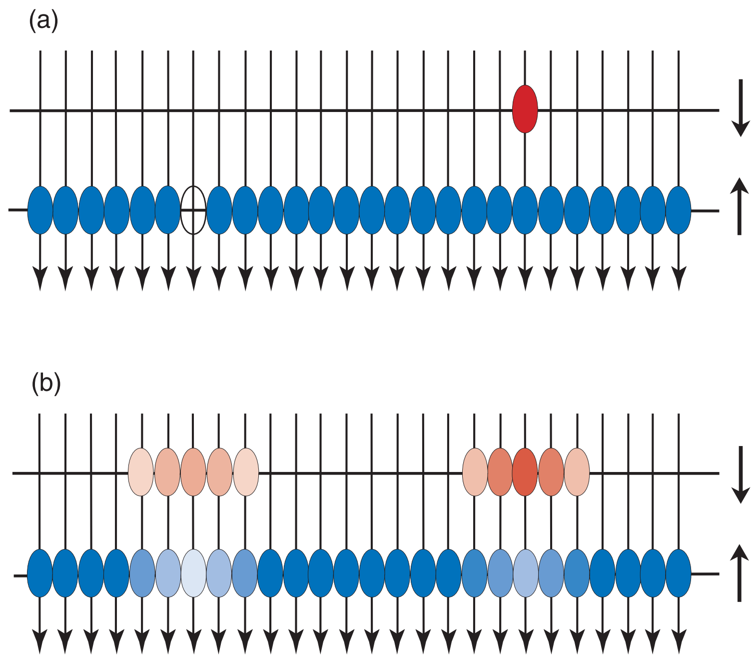

Figure 2: We illustrate the QH ferromagnet at $\nu = 1$. There are one flux quantum per electron. The ground state is a closed packet of electrons pierced by one flux quantum in the spin-up state. (a) When one electron is removed one hole is created, while when one electron is added it is placed in the spin-down state. Their activation energy is $\sim e^2/\varepsilon\ell_B$. (b) To lower the Coulomb energy neighboring electrons make spin rotations The spin rotation modulates the electron density coherently and spreads the charge to a wider domain of size $\kappa\ell_B$, as decreases the Coulomb energy by factor $1/\kappa$. Quasiparticles are coherent excitations called Skyrmions.

by way of a singular phase transformation [14]. We call the resulting electron-flux composite the *bare composite boson*. In order to soften the singularity brought in, we dress it with a cloud of an effective magnetic field that bare composite bosons feel. The resulting object turns out to be the *dressed composite boson*. In Section 3 the relation is established between the electron wave function (1.2) and the composite-boson wave function (1.5). We also verify that the ground state is given by the Laughlin state within the semiclassical approximation. In Section 4 we make a semiclassical analysis of vortex excitations on the basis of the formula (1.6).

In Section 5 we define composite-boson fields with the spin degree of freedom. It is shown that quantum coherence develops spontaneously when the Zeeman effect is small. We review on the Goldstone mode characterizing the QH ferromagnet. In Section 6 we discuss vortex and Skyrmion excitations in the QH ferromagnet. The Skyrmion classical configuration is also derived directly from its microscopic wave function based on the generalized formula of (1.6). We evaluate the activation energy of one Skyrmion and compare it with experimental data. Throughout the paper we use the natural unit $\hbar = c = 1$.

2 Bosonization

The field-theoretical Hamiltonian for spinless planar electrons in external magnetic field $(0, 0, -B)$ is given by

$$H = \frac{1}{2M}\int d^2x\psi^\dagger(\boldsymbol{x})(P_x^2 + P_y^2)\psi(\boldsymbol{x}) + H_C \tag{2.1a}$$

$$= \frac{1}{2M}\int d^2x\psi^\dagger(\boldsymbol{x})(P_x - iP_y)(P_x + iP_y)\psi(\boldsymbol{x}) + \frac{N}{2}\omega_c + H_C, \tag{2.1b}$$

where $\psi(\boldsymbol{x})$ is the electron field; $P_j = -i\partial_j + eA_j^{\text{ext}}$ is the covariant momentum with $A_j^{\text{ext}} = \frac{1}{2}\varepsilon_{jk}x_kB$; N is the electron number and ω_c is the cyclotron frequency. The Coulomb interaction term is

$$H_C = \frac{e^2}{2\varepsilon}\int d^2xd^2y\frac{\varrho(\boldsymbol{x})\varrho(\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|}, \tag{2.2}$$

where $\varrho(\boldsymbol{x}) \equiv \rho(\boldsymbol{x}) - \rho_0$ stands for the deviation of the electron density $\rho(\boldsymbol{x}) \equiv \psi^\dagger(\boldsymbol{x})\psi(\boldsymbol{x})$ from its average value ρ_0.

The state $|\mathfrak{S}\rangle$ in the lowest Landau level obeys

$$(P_x + iP_y)\psi(\boldsymbol{x})|\mathfrak{S}\rangle = -\frac{i}{\ell_B}\Big(z + \frac{\partial}{\partial z^*}\Big)\psi(\boldsymbol{x})|\mathfrak{S}\rangle = 0, \tag{2.3}$$

upon which the kinetic Hamiltonian is trivial. We call it the LLL condition. The generic solution of this equation yields the N-body wave function (1.2) for the state $|\mathfrak{S}\rangle$. We are concerned about the state $|\mathfrak{S}\rangle$ in the lowest Landau level.

We start with a review of the standard bosonization scheme pioneered by Girvin and MacDonald [2]. Its Landau-Ginzburg theory was proposed by Zhang et al. [3, 9] and its microscopic field theory was developped by Ezawa et al. [5, 7]. We define the field $\phi(\boldsymbol{x})$ by an operator phase transformation,

$$\phi(\boldsymbol{x}) = e^{-i\Theta(\boldsymbol{x})}\psi(\boldsymbol{x}), \tag{2.4}$$

where $\Theta(\boldsymbol{x})$ is the phase field,

$$\Theta(\boldsymbol{x}) = m\int d^2y\theta(\boldsymbol{x}-\boldsymbol{y})\rho(\boldsymbol{y}), \tag{2.5}$$

with the azimuthal angle $\theta(\boldsymbol{x}-\boldsymbol{y})$. When m is an odd integer, we can prove that $\phi(\boldsymbol{x})$ is a bosonic operator. Let us call the underlying boson the bare composite boson. It is a hardcore boson satisfying the exclusion principle, $\phi(\boldsymbol{x})^2 = 0$. The LLL condition (2.3) reads

$$(\check{P}_x + i\check{P}_y)\phi(\boldsymbol{x})|\mathfrak{S}\rangle = 0, \tag{2.6}$$

where $\check{P}_j \equiv P_j + \partial_j\Theta(\boldsymbol{x})$ is the covariant momentum for the bare composite-boson field. By solving this condition [5, 16], the N-body wave function is found to be $\mathfrak{S}_\phi[\boldsymbol{x}] \equiv \omega[z]|\mathfrak{S}_{\text{LN}}[\boldsymbol{x}]|$. Namely, the standard bosonization is a mapping from the fermionic wave function $\mathfrak{S}[\boldsymbol{x}]$ to a bosonic wave function $\mathfrak{S}_\phi[\boldsymbol{x}]$,

$$\mathfrak{S}[\boldsymbol{x}] \mapsto \mathfrak{S}_\phi[\boldsymbol{x}] \equiv \omega[z]|\mathfrak{S}_{\text{LN}}[\boldsymbol{x}]|. \tag{2.7}$$

The mapping is to attach m Dirac flux quanta $m\phi_D$ to each electron by way of the phase transformation (2.4), where $\phi_D = 2\pi/e$. Thus, the ground state of bare composite bosons is described by the modulus of the Laughlin function, $|\mathfrak{S}_{\rm LN}[\boldsymbol{x}]|$.

Though the essential physics of the QH effect is revealed by this naive composite-boson theory [2, 3, 5, 8, 9], it is a really complicated theory as the complicated ground-state wave function indicates. Furthermore, the phase transformation (2.4) brings in singularities. In the basis where the electron position is diagonalized, $\rho(\boldsymbol{x}) = \sum_r \delta(\boldsymbol{x} - \boldsymbol{x}_r)$, we obtain

$$e^{i\Theta(\boldsymbol{x})} = \prod_r e^{i\theta(\boldsymbol{x}-\boldsymbol{x}_r)}, \tag{2.8}$$

which is singular at $\boldsymbol{x} = \boldsymbol{x}_r$. These are the reasons why its semiclassical analysis suffers from problems at least in the lowest order approximation. Indeed, it yields the semiclassical ground-state wave function very different from the Laughlin wave function; See (3.14) in a later section.

We wish to introduce another composite-boson field $\varphi(\boldsymbol{x})$ which makes the LLL condition as simple as possible. We set

$$\varphi(\boldsymbol{x}) = e^{-\mathcal{A}(\boldsymbol{x}) - i\Theta(\boldsymbol{x})}\psi(\boldsymbol{x}), \tag{2.9}$$

together with an operator $\mathcal{A}(\boldsymbol{x})$ to be determined later. By substituting this into the LLL condition (2.3), provided $\mathcal{A}(\boldsymbol{x})$ satisfies

$$\partial_j \mathcal{A}(\boldsymbol{x}) = \varepsilon_{jk}\big\{\partial_k\Theta(\boldsymbol{x}) + eA_k(\boldsymbol{x})\big\} = \varepsilon_{jk}\partial_k\Theta(\boldsymbol{x}) - \frac{1}{2\ell_B^2}x_j, \tag{2.10}$$

it is reduced to a simple formula,

$$(\mathcal{P}_x + i\mathcal{P}_y)\varphi(\boldsymbol{x})|\mathfrak{S}\rangle = -\frac{i}{\ell_B}\frac{\partial}{\partial z^*}\varphi(\boldsymbol{x})|\mathfrak{S}\rangle = 0, \tag{2.11}$$

where $\mathcal{P}_j = P_j + \partial_j\Theta(\boldsymbol{x}) + \partial_j\mathcal{A}(\boldsymbol{x})$ is the covariant momentum for the new composite-boson field $\varphi(\boldsymbol{x})$. Eq.(2.10) is easily solved as

$$\mathcal{A}(\boldsymbol{x}) = m\int d^2y \ln\Big(\frac{|\boldsymbol{x}-\boldsymbol{y}|}{2\ell_B}\Big)\rho(\boldsymbol{y}) - |z|^2. \tag{2.12}$$

In the basis with the electron position diagonalized, we find

$$e^{\mathcal{A}(\boldsymbol{x})} \propto \prod_r |\boldsymbol{x} - \boldsymbol{x}_r|^m e^{-|z|^2}, \tag{2.13}$$

which removes the singularities at $\boldsymbol{x} = \boldsymbol{x}_r$ in (2.8).

The N-body wave function $\mathfrak{S}_\varphi[\boldsymbol{x}]$ of dressed composite bosons is obtained by solving the LLL condition (2.11), and it is given simply by an analytic function $\omega[z]$ as in (1.5). We shall verify in the next section that one analytic function $\omega[z]$ characterizes one state $|\mathfrak{S}\rangle$ as in (1.2) in terms of electrons, or as in (2.7) in terms of bare composite bosons, or as in (1.5) in terms of dressed composite bosons. A type of the field operator (2.9) was first considered by Read [4] in constructing a Landau-Ginzburg theory different from the one due to Zhang et al. [3], and revived recently by Rajaraman et al. [17].

The effective magnetic potential for the bare composite boson is $A_k^{\rm ext} + (1/e)\partial_k\Theta$, which is rewritten as $(1/e)\varepsilon_{kj}\partial_j\mathcal{A}(\boldsymbol{x})$. Bare composite bosons feel the effective magnetic field $\mathcal{B}_{\rm eff}(\boldsymbol{x})$,

$$\mathcal{B}_{\rm eff}(\boldsymbol{x}) = e^{-1}\boldsymbol{\nabla}^2\mathcal{A}(\boldsymbol{x}) = m\phi_D\rho(\boldsymbol{x}) - B. \tag{2.14}$$

The effective field vanishes, $\langle \mathcal{B}_{\rm eff}\rangle = 0$, on the homogeneous state $\langle \rho(\boldsymbol{x})\rangle = \rho_0$ if $m\phi_D\rho_0 = B$. It occurs at the filling factor $\nu \equiv \rho_0\phi_D/B = 1/m$. At $\nu = 1/m$ we rewrite the effective magnetic field as

$$e\mathcal{B}_{\rm eff}(\boldsymbol{x}) = \boldsymbol{\nabla}^2\mathcal{A}(\boldsymbol{x}) = 2\pi m\varrho(\boldsymbol{x}), \tag{2.15}$$

with $\varrho(\boldsymbol{y}) \equiv \rho(\boldsymbol{y}) - \rho_0$. It is solved as

$$\mathcal{A}(\boldsymbol{x}) = m\int d^2y \ln\Big(\frac{|\boldsymbol{x}-\boldsymbol{y}|}{2\ell_B}\Big)\varrho(\boldsymbol{y}). \tag{2.16}$$

This formula is equivalent to (2.12) at $\nu = 1/m$. Since the field $\varphi(\boldsymbol{x})$ is constructed by dressing a cloud of the effective magnetic field, we have termed it the dressed composite-boson field.

3 Quantum Hall States

We derive the Hamiltonian in terms of dressed composite bosons by substituting (2.9) into (2.1b),

$$H = \frac{1}{2M}\int d^2x \varphi^\ddagger(\boldsymbol{x})(\mathcal{P}_x - i\mathcal{P}_y)(\mathcal{P}_x + i\mathcal{P}_y)\varphi(\boldsymbol{x}) + \frac{N}{2}\omega_c + H_C, \tag{3.1}$$

where we have defined

$$\varphi^\ddagger(\boldsymbol{x}) \equiv \varphi^\dagger(\boldsymbol{x})e^{2\mathcal{A}(\boldsymbol{x})}, \tag{3.2}$$

with which $\rho(\boldsymbol{x}) = \psi^\dagger(\boldsymbol{x})\psi(\boldsymbol{x}) = \varphi^\ddagger(\boldsymbol{x})\varphi(\boldsymbol{x})$. The covariant momentum $\mathcal{P}_j = P_j + \partial_j\Theta - \partial_j\mathcal{A}$ reads,

$$\mathcal{P}_j = -i\partial_j + e(\delta_{jk} - i\varepsilon_{jk})\mathcal{A}_k, \quad \mathcal{A}_k(\boldsymbol{x}) = -\frac{1}{e}\varepsilon_{kj}\partial_j\mathcal{A}(\boldsymbol{x}). \tag{3.3}$$

The Lagrangian density is

$$\mathcal{L} = \psi^\dagger(i\partial_t - eA_t^{\rm ext})\psi - \mathcal{H} = \varphi^\ddagger(i\partial_t - eA_t^{\rm ext} - \partial_t\Theta - \partial_t\mathcal{A})\varphi - \mathcal{H}, \tag{3.4}$$

where $\mathcal{H}$ is the Hamiltonian density and $A_\mu^{\rm ext} = (A_t^{\rm ext}, A_k^{\rm ext})$ is the potential of the external electromagnetic field. The canonical conjugate of $\varphi(\boldsymbol{x})$ is not $i\varphi^\dagger(\boldsymbol{x})$ but $i\varphi^\ddagger(\boldsymbol{x})$. Hence, the equal-time canonical commutation relations are

$$[\varphi(\boldsymbol{x}), \varphi^\ddagger(\boldsymbol{y})] = \delta(\boldsymbol{x}-\boldsymbol{y}), \qquad [\varphi(\boldsymbol{x}), \varphi(\boldsymbol{y})] = [\varphi^\ddagger(\boldsymbol{x}), \varphi^\ddagger(\boldsymbol{y})] = 0. \tag{3.5}$$

They are also derived [17] by an explicit calculation from those of the electron fields $\psi(\boldsymbol{x})$ and $\psi^\dagger(\boldsymbol{x})$ based on the definition (2.9).

The composite-boson wave function is defined by

$$\mathfrak{S}_\varphi[\boldsymbol{x}] = \langle 0|\varphi(\boldsymbol{x}_1)\varphi(\boldsymbol{x}_2)\cdots\varphi(\boldsymbol{x}_N)|\mathfrak{S}\rangle. \tag{3.6}$$

The LLL condition (2.11) implies that the wave function $\mathfrak{S}_\varphi[\boldsymbol{x}]$ is an analytic function, $\mathfrak{S}_\varphi[\boldsymbol{x}] = \omega[z]$. With the use of the formula (2.12) it is an easy exercise to derive the following relation [17],

$$\varphi^\ddagger(\boldsymbol{x}_1)\varphi^\ddagger(\boldsymbol{x}_2)\cdots\varphi^\ddagger(\boldsymbol{x}_N)|0\rangle = \mathfrak{S}_{\rm LN}[\boldsymbol{x}]\psi^\dagger(\boldsymbol{x}_1)\psi^\dagger(\boldsymbol{x}_2)\cdots\psi^\dagger(\boldsymbol{x}_N)|0\rangle, \tag{3.7}$$

where $\mathfrak{S}_{\rm LN}[\boldsymbol{x}]$ is the Laughlin function (1.3).

Because of the commutation relations (3.5) the state $|\mathfrak{S}\rangle$ associated with the composite-boson wave function (3.6) is given by

$$|\mathfrak{S}\rangle = \int [d\boldsymbol{x}]\mathfrak{S}_\varphi[\boldsymbol{x}]\varphi^\ddagger(\boldsymbol{x}_1)\varphi^\ddagger(\boldsymbol{x}_2)\cdots\varphi^\ddagger(\boldsymbol{x}_N)|0\rangle \tag{3.8}$$

$$= \int [d\boldsymbol{x}]\omega[z]\mathfrak{S}_{\rm LN}[\boldsymbol{x}]\psi^\dagger(\boldsymbol{x}_1)\psi^\dagger(\boldsymbol{x}_2)\cdots\psi^\dagger(\boldsymbol{x}_N)|0\rangle, \tag{3.9}$$

where use was made of (3.7), and $[d\boldsymbol{x}] = d^2x_1 d^2x_2\cdots d^2x_N$. It follows from (3.9) that the electron wave function is $\mathfrak{S}[\boldsymbol{x}] = \omega[z]\mathfrak{S}_{\rm LN}[\boldsymbol{x}]$, as verifies the basic formulas (1.2) $\sim$ (1.5).

One might question the hermiticity of the theory [17] since the covariant momentum (3.3) has an unusual expression. It is related to the fact that the canonical conjugate of $\varphi(\boldsymbol{x})$ is $i\varphi^\ddagger(\boldsymbol{x}) \equiv i\varphi^\dagger(\boldsymbol{x})e^{2\mathcal{A}(\boldsymbol{x})}$. It implies that the hermiticity is defined together with the measure $e^{2\mathcal{A}(\boldsymbol{x})}$. Such a measure has arisen since the transformation (2.9) is not unitary. It is trivial to check explicitly that the covariant momentum $\mathcal{P}_j$ is hermitian together with this measure. It is also instructive to rewrite the Hamiltonian (3.1) as

$$H = \frac{\omega_c}{2}\int d^2x\Big(\frac{\partial}{\partial z^*}\varphi(\boldsymbol{x})\Big)^\dagger e^{2\mathcal{A}(\boldsymbol{x})}\frac{\partial}{\partial z^*}\varphi(\boldsymbol{x}) + \frac{N}{2}\omega_c + H_C, \tag{3.10}$$

which is manifestly hermitian.

The semiclassical ground state is the one that minimizes the total energy $\langle H\rangle$. The Coulomb energy (2.2) is minimized by the state where $\langle\varrho(\boldsymbol{x})\rangle = 0$. It is realized when the electron density is homogeneous,

$$\langle\rho(\boldsymbol{x})\rangle = \langle\varphi^\ddagger(\boldsymbol{x})\varphi(\boldsymbol{x})\rangle = e^{2\langle\mathcal{A}(\boldsymbol{x})\rangle}\langle\varphi^\dagger(\boldsymbol{x})\varphi(\boldsymbol{x})\rangle = \rho_0. \tag{3.11}$$

In the semiclassical approximation we obtain

$$\prod_{r=1}^N\langle\rho(\boldsymbol{x}_r)\rangle = \prod_{r=1}^N e^{2\langle\mathcal{A}(\boldsymbol{x}_r)\rangle}\langle\mathfrak{S}|\varphi^\dagger(\boldsymbol{x}_N)\cdots\varphi^\dagger(\boldsymbol{x}_2)\varphi^\dagger(\boldsymbol{x}_1)\varphi(\boldsymbol{x}_1)\varphi(\boldsymbol{x}_2)\cdots\varphi(\boldsymbol{x}_N)|\mathfrak{S}\rangle. \tag{3.12}$$

We insert a complete set $\sum|n\rangle\langle n| = 1$ between two operators $\varphi^\dagger(\boldsymbol{x}_1)$ and $\varphi(\boldsymbol{x}_1)$. When the state $|\mathfrak{S}\rangle$ contains N electrons, only the vacuum term $|0\rangle\langle 0|$ survives in the complete set because $\varphi(\boldsymbol{x}_r)$ decreases the electron number by one. Hence, N-body wave function (3.6) is given by

$$\mathfrak{S}_\varphi[\boldsymbol{x}] = \rho_0^{N/2}\prod_{r=1}^N e^{-\langle\mathcal{A}(\boldsymbol{x}_r)\rangle}, \tag{3.13}$$

up to an irrelevant phase factor. On the other hand, to suppress the kinetic energy we impose the LLL condition (2.11), as requires $\mathfrak{S}_\varphi[\boldsymbol{x}]$ to be analytic. Consequently, it follows that $\langle\mathcal{A}(\boldsymbol{x})\rangle =$ constant or $\mathcal{B}_{\rm eff} = 0$, which is possible only at $\nu = 1/m$ from (2.14). Namely, the ground state is realized only at the magic filling factor $\nu = 1/m$. At $\nu = 1/m$ the ground-state wave function is given by $\mathfrak{S}_\varphi[\boldsymbol{x}] =$ constant in terms of composite bosons, and therefore by the Laughlin wave function (1.3) in terms of electrons. In this way the Laughlin state is proved to be the ground state in the improved composite-boson theory. The ground state is illustrated in Fig.1(a).

For the sake of completeness, we briefly recall the results [5] of the corresponding analysis with use of bare composite bosons. We can derive the equations similar to (3.8) and (3.9), which verifies the mapping (2.7). The semiclassical ground state is similarly given by $\mathfrak{S}_\phi(\boldsymbol{x}) =$ constant in terms of bare composite bosons. The wave function in terms of electrons is singular,

$$\mathfrak{S}[\boldsymbol{x}] = e^{im\sum_{r<s}\theta(z_r - z_s)}. \tag{3.14}$$

The state does not belong to the lowest Landau level. Although the singular short-distance behavior in (3.14) is remedied in a higher order perturbation theory [5, 7], it makes the naive composite-boson theory less attractive.

4 Semiclassical Analysis

We analyze excitations on the QH state. A priori two types of excitations are possible, that is, perturbative and nonperturbative ones in terms of the density fluctuation $\varrho(\boldsymbol{x})$ and its conjugate phase $\chi(\boldsymbol{x})$. A perturbative analysis has already been carried out based on the bare composite-boson theory [5]. The result is that there exist no perturbative fluctuations confined within the lowest Landau level. This conclusion remains to be true in the improved theory. To show it, we parametrize the bare field as $\phi(\boldsymbol{x}) = e^{i\chi(\boldsymbol{x})}\sqrt{\rho_0 + \varrho(\boldsymbol{x})}$ in terms of the density deviation $\varrho(\boldsymbol{x})$ and its canonical phase $\chi(\boldsymbol{x})$. The dressed field $\varphi(\boldsymbol{x})$ is a nonlocal operator due to the factor $e^{-\mathcal{A}(\boldsymbol{x})}$ with (2.16),

$$\varphi(\boldsymbol{x}) = e^{-\mathcal{A}(\boldsymbol{x})}e^{i\chi(\boldsymbol{x})}\sqrt{\rho_0 + \varrho(\boldsymbol{x})}. \tag{4.1}$$

Substituting (4.1) into the Hamiltonian (3.1) and expanding various quantities in term of $\varrho(\boldsymbol{x})$ and $\chi(\boldsymbol{x})$, the perturbative Hamiltonian is found to be identical between the bare and dressed composite-boson theory, as should be the case. We conclude that all excitations in the lowest Landau level are nonperturbative objects.

The improved theory confirms this assertion by showing explicitly how they are created on the ground state. Indeed, any excited state is represented as in (3.8), which is a nonlocal object because the creation operator $\varphi^{\ddagger}(\boldsymbol{x})$ is a nonlocal operator as in (4.1).

4.1 Vortices

We first examine excited states when the wave function (3.6) is factorizable, $\mathfrak{S}_\varphi[\boldsymbol{x}] = \omega[z] = \prod_r \omega(z_r)$. In this case we can easily make a semiclassical analysis of the one-point function $\langle\varphi(\boldsymbol{x})\rangle = \omega(z)$ by setting

$$e^{-\mathcal{A}(\boldsymbol{x})}e^{i\chi(\boldsymbol{x})}\sqrt{\rho_0 + \varrho(\boldsymbol{x})} = \omega(z), \tag{4.2}$$

based on the parametrization (4.1). Here and hereafter, we use the same symbols $\mathcal{A}(\boldsymbol{x})$, $\varrho(\boldsymbol{x})$ and $\chi(\boldsymbol{x})$ also for the classical fields. When an analytic function $\omega(z)$ is given, (4.2) is an integral equation determining the density deviation $\varrho(\boldsymbol{x})$ confined within the lowest Landau level.

We transform (4.2) into a differential equation. The Cauchy-Riemann equation for the analytic function (4.2) yields,

$$\partial_j\big(\mathcal{A}(\boldsymbol{x}) - \ln\sqrt{\rho_0 + \varrho(\boldsymbol{x})}\big) = -\varepsilon_{jk}\partial_k\chi(\boldsymbol{x}). \tag{4.3}$$

Using (2.15) we obtain that

$$\frac{\nu}{4\pi}\boldsymbol{\nabla}^2\ln\Big(1 + \frac{\varrho(\boldsymbol{x})}{\rho_0}\Big) - \varrho(\boldsymbol{x}) = \nu Q_0^V(\boldsymbol{x}), \tag{4.4}$$

which we call the soliton equation, where

$$Q_0^V(\boldsymbol{x}) = \frac{1}{2\pi}\varepsilon_{jk}\partial_j\partial_k\chi(\boldsymbol{x}) = \frac{1}{2\pi i}\varepsilon_{jk}\partial_j\partial_k\ln\omega(z) \tag{4.5}$$

is the topological charge density associated with the excitation. This is nonvanishing since $\ln \omega(z)$ is a multivalued function unless $\omega(z) = \text{constant}$. The topological current is

$$Q_\mu^V(\boldsymbol{x}) = \frac{1}{2\pi}\varepsilon_{\mu\nu\lambda}\partial_\nu\partial_\lambda\chi(\boldsymbol{x}), \tag{4.6}$$

which is a conserved quantitity, $\partial^\mu Q_\mu^V(\boldsymbol{x}) = 0$. Topological solitons are generated around the zeros of $\omega(z)$ according to the soliton equation (4.4). The density modulation is induced in order to confine the excitation within the lowest Landau level.

The topological charge is evaluated as

$$Q^V = \int d^2x Q_0^V(\boldsymbol{x}) = \frac{1}{2\pi i}\oint dx_k\partial_k \ln \omega(z), \tag{4.7}$$

where the loop integration $\oint$ is made to encircle the excitation at infinity ($|\boldsymbol{x}| \to \infty$) provided $\omega(z)$ is regular everywhere. The topological charge is uniquely determined by the asymptotic behavior of the classical field $\varphi(\boldsymbol{x})$,

$$\varphi(\boldsymbol{x}) \to \sqrt{\rho_0}z^q, \qquad \text{as} \quad |z| \to \infty, \tag{4.8}$$

for which the electron number is

$$\Delta N = \int d^2x\varrho(\boldsymbol{x}) = -\nu Q_{\mathrm{V}} = -\nu q, \tag{4.9}$$

as follows from the soliton equation (4.4). The electric charge carried by this soliton is $-e\Delta N = \nu q e$. It is a hole made in the condensate of composite bosons, as is illustrated for the $q = 1$ vortex in Fig.1(c). We may as well derive the electric number (4.9) more directly from the parametrization (4.2). We take the asymptotic behavior $|z| \to \infty$ in (4.2) and equate it with (4.8),

$$\varrho(\boldsymbol{x}) \to 0, \qquad \chi(\boldsymbol{x}) \to q\theta, \qquad \mathcal{A}(\boldsymbol{x}) \to -q\ln|z|, \qquad \text{as} \quad |z| \to \infty. \tag{4.10}$$

Taking the limit $|\boldsymbol{x}| \to \infty$ in (2.16) we recover the quantization of the electron number (4.9). Actually we have determined the coefficient $\sqrt{\rho_0}$ in the asymptotic behavior (4.8) in this way.

A topological excitation carries a quantized topological charge, which has to be created all at once. It cannot be created by an accumulation of perturbative fluctuations, as agrees with the perturbative result mentioned at the begining of this section. In the absence of the Coulomb term all these excitations are degenerate with the ground state, which explains the degeneracy in the lowest Landau level at $\nu < 1$. The degeneracy is removed since any density moduration acquires a Coulomb energy,

$$\langle H_C\rangle = \frac{e^2}{2\varepsilon}\int d^2xd^2y\frac{\varrho(\boldsymbol{x})\varrho(\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|} = \alpha_C\nu^2q^2\frac{e^2}{\varepsilon\ell_B}, \tag{4.11}$$

where α_C is a constant of order one.

An explicit example is given by a vortex (quasihole) sitting at $\boldsymbol{x} = 0$, whose wave function is $\mathfrak{S}_\varphi[\boldsymbol{x}] = \prod_r z_r$ up to a normalization factor, or $\omega(z) = \sqrt{\rho_0}z$. In this example the topological charge (4.5) is concentrated at the vortex center, $Q_0^V(\boldsymbol{x}) = \delta(\boldsymbol{x})$. A crude approximation of the soliton equation (4.4) reads

$$\frac{\ell_B^2}{2}\boldsymbol{\nabla}^2\varrho(\boldsymbol{x}) - \varrho(\boldsymbol{x}) = \nu\delta(\boldsymbol{x}), \tag{4.12}$$

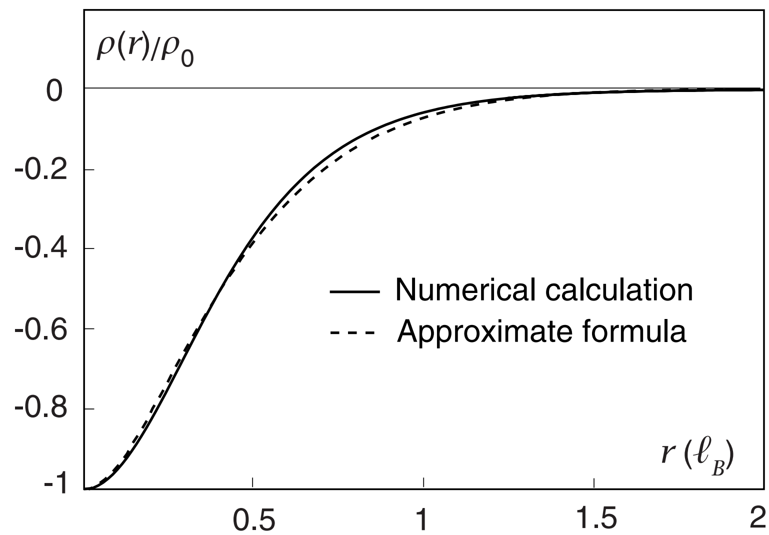

Figure 3: The density moduration around a vortex with $q = 1$ is plotted. The solid curve is obtained by solving the differential equation (4.14) numerically. The dashed curve is drawn by using the approximate formula (4.13).

since $\nu = 2\pi\rho_0\ell_B^2$. Its exact solution [6] is $\varrho(\boldsymbol{x}) = -(\nu/\pi\ell_B^2)K_0(s)$ with $s = \sqrt{2}r/\ell_B$ and $K_0(s)$ the modified Bessel function. This is a rather poor approximation because of its singular behavior, $\varrho(\boldsymbol{x}) \to -\infty$, at the vortex center ($\boldsymbol{x} = 0$). A better approximation is given by

$$\varrho(\boldsymbol{x}) = -\rho_0\Big(1 + s - \frac{s^2}{6}\Big)e^{-s}, \tag{4.13}$$

which has the correct behavior both at $\boldsymbol{x} = 0$ and $|\boldsymbol{x}| \gg \ell_B$. Furthermore, it has the correct topological charge. For a numerical analysis, it is convenient to set $\rho(\boldsymbol{x}) = \rho_0 e^{u(s)}$ in the soliton equation (4.4), as yields

$$\frac{d^2u}{ds^2} + \frac{1}{s}\frac{du}{ds} + 1 = e^{u(s)}, \tag{4.14}$$

for $s > 0$. The result of a numerical analysis shows that the density modulation is well approximated by (4.13), as in Fig.4.1. The Coulomb energy is given by (4.11) with $\alpha_C \simeq 0.39$.

4.2 Antivortices

We have so far considered the case where the N-point function is factorizable, $\omega[z] = \prod_r \omega(z_r)$. A vortex is described by the choice of $\omega[z] = \prod_r z_r$ up to a normalization, which gives an angular momentum to each electron. Similarly, an antivortex is generated by decreasing an angular momentum from each electron. Naively, this is to multiply z_r^* to the rth electron, but we cannot accept this option because the resulting configuration does not stay within the lowest Landau level. The only possible way seems to use the derivative operation $\prod \partial/\partial z_r$ to decrease the angular momentum [6]. It is convenient to write symbolically as

$$\omega[z] = \prod_r \frac{\partial}{\partial z_r}, \tag{4.15}$$

up to a normalization, with the understanding that it acts only on the polynomial part of the wave function (1.2). Then, the wave function $\mathfrak{S}_{\varphi}[\boldsymbol{x}]$ remains to be analytic but becomes very complicated. It is no longer factorizable, as makes the analysis of antivortices considerably complicated.

We analyze a generic excitation on the Laughlin state. We propose to define one-point function by the formula,

$$\langle\varphi(\boldsymbol{x})\rangle = \lim_{N\to\infty}\langle\mathfrak{S}^{N}|\varphi(\boldsymbol{x})|\mathfrak{S}^{N+1}\rangle, \tag{4.16}$$

by taking two states $|\mathfrak{S}^{N}\rangle$ and $|\mathfrak{S}^{N+1}\rangle$ which are identical except the number of electrons in the ground state. Let us give an example of a single vortex sitting at the center of a large system. Then, these two states are defined by the wave functions $\mathfrak{S}^{N}[\boldsymbol{x}] = \prod_{r=1}^{N} z_r \mathfrak{S}_{\text{LN}}^{N}[\boldsymbol{x}]$ and $\mathfrak{S}^{N+1}[\boldsymbol{x}] = \prod_{r=1}^{N+1} z_r \mathfrak{S}_{\text{LN}}^{N+1}[\boldsymbol{x}]$, where $\mathfrak{S}_{\text{LN}}^{N}[\boldsymbol{x}]$ is the Laughlin wave function containing N electrons. The semiclassical property of the vortex is independent of the number N of electrons in the system provided N is large.

Using (3.8) and (3.9) in (4.16), we may express $\langle\varphi(\boldsymbol{x})\rangle$ as

$$\langle\varphi(\boldsymbol{x})\rangle = \int [d\boldsymbol{x}]\omega(z_1, z_2, \cdots, z_N)^{*}\omega(z_1, z_2, \cdots, z_N, z)|\mathfrak{S}_{\text{LN}}^{N}[\boldsymbol{x}]|^2. \tag{4.17}$$

This is not an analytic function in general, although we recover $\langle\varphi(\boldsymbol{x})\rangle = \omega(z)$ when factorizable, $\omega[z] = \prod_r \omega(z_r)$. Nevertheless, by inspecting the integration (4.17), we may conclude that it becomes an analytic function asymptotically, $\langle\varphi(\boldsymbol{x})\rangle \to \omega(z)$ as $|z| \to \infty$, since the integration over $\boldsymbol{x}_r$ is convergent due to the gaussian factor in the Laughlin function. See (4.22) and (4.23) explicitly.

Although the Cauchy-Rieman equation (4.3) does not follow, we obtain the soliton equation (4.4) with

$$Q_0^V(\boldsymbol{x}) = \frac{1}{2\pi}\boldsymbol{\nabla}^2(\ln\langle\varphi\rangle - i\chi). \tag{4.18}$$

In evaluating the electron number $\int d^2x\varrho(\boldsymbol{x})$, therefore, only the bounday value of $\langle\varphi(\boldsymbol{x})\rangle$ is relevant, where it approaches an analytic function $\omega(z)$. The electron number is given by the same formula as (4.9) in terms of the asymptotic behavior of $\langle\varphi(\boldsymbol{x})\rangle$.

We give an example for the antivortex (quasielectron). Evaluating the polynomial part in (1.2) with (4.15) explicitly, the N-body wave function $\omega_{qe}[z]$ for an antivortex sitting at the origin is given by way of the formula [6]

$$\prod_t \frac{\partial}{\partial z_t}\prod_{r<s}(z_r - z_s)^m \equiv \omega_{qe}[z]\prod_{r<s}(z_r - z_s)^m, \tag{4.19}$$

or

$$\mathfrak{S}_{\varphi}^{qe}[z] = \omega_{qe}[z] = \frac{\prod_t \frac{\partial}{\partial z_t}\prod_{r<s}(z_r - z_s)^m}{\prod_{r<s}(z_r - z_s)^m}. \tag{4.20}$$

This is the wave function of a quasielectron in terms of composite bosons. The one-point function (4.17) is

$$\langle\varphi_{qe}(\boldsymbol{x})\rangle = \int [d\boldsymbol{x}]\omega_{qe}(z_1, z_2, \cdots, z_N)^{*}\omega_{qe}(z_1, z_2, \cdots, z_N, z)|\mathfrak{S}_{\text{LN}}^{N}[\boldsymbol{x}]|^2. \tag{4.21}$$

The dominant term for $|z| \gg 1$ is given by

$$\langle \varphi_{qe}(\boldsymbol{x}) \rangle \simeq G(\boldsymbol{x}) = \sum_{s=1}^{N} \int [d\boldsymbol{x}] \frac{1}{z - z_s} |\mathfrak{S}_{qe}^{N}(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_N)|^2, \tag{4.22}$$

up to a normalization factor, where $\mathfrak{S}_{qe}^{N} = \omega_{qe}\mathfrak{S}_{\rm LN}^{N}$ is the N-body quasielectron wave function. We calculate $G(\boldsymbol{x})$ in the domain $|\boldsymbol{x}| \gg 1$, where we may expand $(z - z_s)^{-1}$ in a power series of z_s/z, and find $G(\boldsymbol{x}) = z^{-1}$. If it were an analytic function, we would have $G(\boldsymbol{x}) = z^{-1}$ by analytic continuation. However, since $G(0) = 0$ at $z = 0$ by integrating over the angle variable, it cannot be so, though it approaches an analytic function asymptotically.

We give an explicit example of the function of type (4.22),

$$G_1(\boldsymbol{x}) = \frac{1}{\pi} \int \frac{d^2\omega}{z - \omega} e^{-|\omega|^2} = \frac{1}{z}\Big(1 - e^{-|z|^2}\Big). \tag{4.23}$$

We explicitly see that $G_1(\boldsymbol{x}) \to z^{-1}$ exponentially fast for $|z| \gg 1$ and yet $G_1(\boldsymbol{x}) = 0$ at $\boldsymbol{x} = 0$.

We may in principle determine the classical configuration of the antivortex from (4.1) once we obtain a closed formula for (4.21) in the limit $N \to \infty$, which is yet to be done. However, it is clear that $\langle \varphi(\boldsymbol{x}) \rangle$ approaches its asymptotic value z^{-1} exponentially fast outside the core of size $\sim \ell_B$. The antivortex is a lump of a quantized electric charge $-e/m$ in the homogeneous ground state as illustrated in Fig.1(d). We expect that the activation energy of one antivortex is nearly the same as that of one vortex. In general we have the asymptotic behavior (4.8) for vortex ($q > 0$) and antivortex ($q < 0$) excitations. It characterizes topological excitations on the Laughlin state, as we have noticed in (4.10). Thirmal fluctuations generate vortex-antivortex pairs as in Fig.1(e), which is ditected by measuring the longitudinal resistivity (1.1).

5 Quantum Hall Ferromagnet

We proceed to analyze the QH system with the SU(2) symmetry. The electron field $\psi^\alpha(\boldsymbol{x})$ has the index $\alpha =\uparrow, \downarrow$. It denotes the electron spin in the monolayer QH system with the spin SU(2) symmetry, or the layer index in a certain bilayer QH system with the pseudospin SU(2) symmetry. For definiteness we analyze the monolayer spin system in this paper.

The Hamiltonian depends on the electron spin through the Zeeman energy term,

$$H_Z = -g^* \mu_B B \int d^2x S^z(\boldsymbol{x}), \tag{5.1}$$

with $S^z = \frac{1}{2}(\psi^{\uparrow\dagger}\psi^{\uparrow} - \psi^{\downarrow\dagger}\psi^{\downarrow})$, where g^* is the gyromagnetic factor and μ_B the Bohr magneton. Each Landau level contains two energy levels with the one-particle gap energy $g^*\mu_B B$. The lowest Landau level is filled at $\nu = 2$. We consider the case where the Zeeman energy is much smaller than the Coulomb energy. Though one Landau level contains two degenerate energy levels in the vanishing limit of the Zeeman coupling ($g^* = 0$), the system becomes incompressible at $\nu = 1/m$. The physical reason is the Coulomb exchange energy, as we now discuss.

We define the *bare* composite-boson field $\phi^\alpha(\boldsymbol{x})$ and the *dressed* composite-boson field $\varphi^\alpha(\boldsymbol{x})$ by

$$\phi^\alpha(\boldsymbol{x}) = e^{-i\Theta(\boldsymbol{x})}\psi^\alpha(\boldsymbol{x}), \quad \varphi^\alpha(\boldsymbol{x}) = e^{-\mathcal{A}(\boldsymbol{x})}\phi^\alpha(\boldsymbol{x}), \tag{5.2}$$

where the phase field $\Theta(\boldsymbol{x})$ and the auxiliary field $\mathcal{A}(\boldsymbol{x})$ are given by (2.5) and (2.16), respectively, with the total electron density $\rho(\boldsymbol{x}) = \sum_\alpha \psi^{\alpha\dagger}(\boldsymbol{x})\psi^\alpha(\boldsymbol{x}) = \sum_\alpha \varphi^{\alpha\dagger}(\boldsymbol{x})e^{2\mathcal{A}(\boldsymbol{x})}\varphi^\alpha(\boldsymbol{x})$. The Hamiltonian is

$$H = \frac{1}{2M}\sum_\alpha \int d^2x \psi^{\alpha\dagger}(\boldsymbol{x})(P_x^2 + P_y^2)\psi^\alpha(\boldsymbol{x}) + H_C + H_Z \tag{5.3}$$

$$= \omega_c \sum_\alpha \int d^2x \Big(\frac{\partial}{\partial z^*}\varphi^\alpha(\boldsymbol{x})\Big)^\dagger e^{2\mathcal{A}(\boldsymbol{x})}\frac{\partial}{\partial z^*}\varphi^\alpha(\boldsymbol{x}) + \frac{N}{2}\omega_c + H_C + H_Z, \tag{5.4}$$

with the Coulomb term H_C and the Zeeman term H_Z. The Coulomb term depends on the deviation $\varrho(\boldsymbol{x})$ of the total electron density from the average density, $\varrho(\boldsymbol{x}) = \rho(\boldsymbol{x}) - \rho_0$, as in (2.2).

5.1 Spin Texture

We may decompose the bare composite-boson field into the U(1) field $\phi(\boldsymbol{x})$ and the SU(2) field $n^\alpha(\boldsymbol{x})$,

$$\phi^\alpha(\boldsymbol{x}) = \phi(\boldsymbol{x})n^\alpha(\boldsymbol{x}), \quad \sum_\alpha n^{\alpha\dagger}n^\alpha = 1. \tag{5.5}$$

The field $n^\alpha(\boldsymbol{x})$ is the complex-projective (CP^1) field[18], whose overall phase has been removed and given to the U(1) field $\phi(\boldsymbol{x})$. The spin operator is expressed as

$$S^a(\boldsymbol{x}) = \frac{1}{2}\rho(\boldsymbol{x})\Sigma^a(\boldsymbol{x}), \tag{5.6}$$

where

$$\Sigma^a(\boldsymbol{x}) = \boldsymbol{n}^\dagger(\boldsymbol{x})\tau^a\boldsymbol{n}(\boldsymbol{x}), \quad \boldsymbol{n}(\boldsymbol{x}) = \begin{pmatrix} n^\uparrow(\boldsymbol{x}) \\ n^\downarrow(\boldsymbol{x}) \end{pmatrix}. \tag{5.7}$$

In terms of the dressed composite-boson field the decomposition reads

$$\varphi^\alpha(\boldsymbol{x}) = \varphi(\boldsymbol{x})n^\alpha(\boldsymbol{x}), \quad \varphi(\boldsymbol{x}) = e^{-\mathcal{A}(\boldsymbol{x})}\phi(\boldsymbol{x}). \tag{5.8}$$

The SU(2) component $n^\alpha(\boldsymbol{x})$ is common between the bare and dressed fields (5.5) and (5.8): It is a local field. On the other hand, $\varphi(\boldsymbol{x})$ is a nonlocal field due to the factor $e^{-\mathcal{A}(\boldsymbol{x})}$ as in the spinless theory.

The ground state minimizes both the Coulomb and Zeeman energies. The Coulomb energy is minimized by the homogeneous electron density, $\langle\rho(\boldsymbol{x})\rangle = \rho_0$. The Zeeman energy is minimized when all electrons are polarized into the positive z axis, $\langle n^\uparrow(\boldsymbol{x})\rangle = 1$ and $\langle n^\downarrow(\boldsymbol{x})\rangle = 0$. The ground state is unique, which we denote by $|g_0\rangle$.

We consider a spin texture given by performing an SU(2) transformation on the ground state $|g_0\rangle$, $|\widetilde{\mathfrak{S}}\rangle = e^{i\mathcal{O}}|g_0\rangle$, where $\mathcal{O}$ is its generator,

$$\mathcal{O} = \sum_a \int d^2x f^a(\boldsymbol{x})S^a(\boldsymbol{x}) = \sum_a \int d^2q f^a_{-\boldsymbol{q}}S^a_{\boldsymbol{q}}. \tag{5.9}$$

It is described by the classical sigma field $s^a(\boldsymbol{x}) = \langle\widetilde{\mathfrak{S}}|\Sigma^a(\boldsymbol{x})|\widetilde{\mathfrak{S}}\rangle$, which we parametrize as

$$s^x(\boldsymbol{x}) = \sigma(\boldsymbol{x}), \quad s^y(\boldsymbol{x}) = \sqrt{1-\sigma^2(\boldsymbol{x})}\sin\vartheta(\boldsymbol{x}), \quad s^z(\boldsymbol{x}) = \sqrt{1-\sigma^2(\boldsymbol{x})}\cos\vartheta(\boldsymbol{x}). \tag{5.10}$$

The spin texture is classified by the Pontryagin number [18], $Q^P = \int d^2x Q_0^P(\boldsymbol{x})$, with the topological current

$$Q_\mu^P(\boldsymbol{x}) = \frac{1}{8\pi}\varepsilon_{abc}\varepsilon_{\mu\nu\lambda}s_a\partial^\nu s_b\partial^\lambda s_c. \tag{5.11}$$

It is absolutely conserved, $\partial^\mu Q_\mu^P = 0$. We note that the spin texture $|\widetilde{\mathfrak{S}}\rangle$ does not belong to the lowest Landau level.

5.2 Wave Functions

The two-component composite-boson field is $\Phi(\boldsymbol{x}) = \varphi(\boldsymbol{x})\boldsymbol{n}(\boldsymbol{x})$. With the Hamiltonian (5.4), the LLL condition for the state $|\mathfrak{S}\rangle$ is

$$\frac{\partial}{\partial z^*}\Phi(\boldsymbol{x})|\mathfrak{S}\rangle = 0. \tag{5.12}$$

Because of this condition the N-body wave function is analytic,

$$\mathfrak{S}_\varphi[\boldsymbol{x}] = \langle 0|\Phi(\boldsymbol{x}_1)\Phi(\boldsymbol{x}_2)\cdots\Phi(\boldsymbol{x}_N)|\mathfrak{S}\rangle = \Omega[z], \tag{5.13}$$

where $\Omega[z]$ is totally symmetric in N variables. When it is factorizable, $\Omega[z] = \prod_r \Omega(z_r)$, it has a simple expression,

$$\Omega(z) = \begin{pmatrix}\omega^\uparrow(z)\\ \omega^\downarrow(z)\end{pmatrix}. \tag{5.14}$$

For the ground state $|g_0\rangle$ we have

$$\Omega(z) = \sqrt{\rho_0}\begin{pmatrix}1\\ 0\end{pmatrix}. \tag{5.15}$$

In terms of the original electrons the wave function is given by

$$\Psi_\varphi[z, z^*] = \Omega[z]\mathfrak{S}_{\rm LN}[\boldsymbol{x}], \tag{5.16}$$

with the Laughlin wave function $\mathfrak{S}_{\rm LN}[\boldsymbol{x}]$.

5.3 LLL Projection

The energy of the spin texture $\langle\widetilde{\mathfrak{S}}|H|\widetilde{\mathfrak{S}}\rangle$ acquires a contribution from the kinetic Hamiltonian, which is of the order of the Landau-level gap energy $\hbar\omega_c$. This is because the spin texture $|\widetilde{\mathfrak{S}}\rangle$ does not belong to the lowest Landau level. It is necessary to excite only the component $|\mathfrak{S}\rangle$ belonging to the lowest Landau level by requiring (5.12). We have two complementary methods. One method is to solve the LLL condition (5.12) semiclassically and determine the density modulation as well as the spin modulation, as is the method we have used to analyze the vortex excitation in the previous section. We use the same method to analyze the Skyrmion excitation in the following section. The other is an algebraic method based on the LLL projection [13]. Here, we use it to derive the effective Hamiltonian govering a small spin fluctuation around the ground state.

The LLL component $|\mathfrak{S}\rangle$ is extracted from the spin texture $|\widetilde{\mathfrak{S}}\rangle$ by extracting the LLL component $\langle\!\langle f^a(\boldsymbol{x})\rangle\!\rangle$ from the "wave packet" $f^a(\boldsymbol{x})$ in the generator $\mathcal{O}$ of the SU(2) transformation (5.9). This turns out to replace the plane weve $e^{i\boldsymbol{xq}}$ with [8]

$$\langle\!\langle e^{i\boldsymbol{xq}}\rangle\!\rangle \equiv e^{-\frac{1}{4}\boldsymbol{q}^2\ell_B^2}e^{i\boldsymbol{Xq}} \tag{5.17}$$

in the Fourier representation of $f^a(\boldsymbol{x})$, where $\boldsymbol{X} = (X, Y)$ is the guiding center. We call $\langle\!\langle e^{i\boldsymbol{xq}}\rangle\!\rangle$ the LLL projection of $e^{i\boldsymbol{xq}}$. The generator (5.9) is projected as

$$\widehat{O} = \sum_a \int d^2x \langle\!\langle f^a(\boldsymbol{x})\rangle\!\rangle S^a(\boldsymbol{x}) = \sum_a \int d^2q f^a_{-\boldsymbol{q}} \widehat{S}^a_{\boldsymbol{q}}, \tag{5.18}$$

where

$$\widehat{S}^a_{\boldsymbol{q}} = (2\pi)^{-1} \int d^2x \langle\!\langle e^{-i\boldsymbol{qx}}\rangle\!\rangle S^a(\boldsymbol{x}). \tag{5.19}$$

Similarly we define

$$\widehat{\rho}_{\boldsymbol{q}} = (2\pi)^{-1} \int d^2x \langle\!\langle e^{-i\boldsymbol{qx}}\rangle\!\rangle \rho(\boldsymbol{x}) \tag{5.20}$$

for the electron density operator.

From the commutation relation $[X, Y] = -i\ell_B^2$ between the X and Y components of the guiding center, we obtain

$$[e^{i\boldsymbol{qX}}, e^{i\boldsymbol{pX}}] = 2ie^{i(\boldsymbol{q}+\boldsymbol{p})\boldsymbol{X}} \sin\big[\ell_B^2 \frac{\boldsymbol{q}\wedge\boldsymbol{p}}{2}\big], \tag{5.21}$$

with $\boldsymbol{q}\wedge\boldsymbol{p} \equiv q_x p_y - q_y p_x$. The translation $e^{i\boldsymbol{qx}}$ is Abelian, but the magnetic translation $e^{i\boldsymbol{qX}}$ is non-Abelian. It governs the symmetric structure of the two-dimensional space after the LLL projection. It is straightforward to derive the following $\mathrm{W}_\infty\times\mathrm{SU}(2)$ algebra [8],

$$[\widehat{\rho}_{\boldsymbol{p}}, \widehat{\rho}_{\boldsymbol{q}}] = \frac{i}{\pi}\widehat{\rho}_{\boldsymbol{p}+\boldsymbol{q}} \sin\big[\ell_B^2 \frac{\boldsymbol{p}\wedge\boldsymbol{q}}{2}\big] \exp\big[\frac{\ell_B^2}{2}\boldsymbol{pq}\big], \tag{5.22a}$$

$$[\widehat{S}^a_{\boldsymbol{p}}, \widehat{\rho}_{\boldsymbol{q}}] = \frac{i}{\pi}\widehat{S}^a_{\boldsymbol{p}+\boldsymbol{q}} \sin\big[\ell_B^2 \frac{\boldsymbol{p}\wedge\boldsymbol{q}}{2}\big] \exp\big[\frac{\ell_B^2}{2}\boldsymbol{pq}\big], \tag{5.22b}$$

$$\begin{aligned}[\widehat{S}^a_{\boldsymbol{p}}, \widehat{S}^b_{\boldsymbol{q}}] = {} & \frac{i}{2\pi}\varepsilon^{abc}\widehat{S}^c_{\boldsymbol{p}+\boldsymbol{q}} \cos\big[\ell_B^2 \frac{\boldsymbol{p}\wedge\boldsymbol{q}}{2}\big] \exp\big[\frac{\ell_B^2}{2}\boldsymbol{pq}\big] \\ & + \frac{i}{4\pi}\delta^{ab}\widehat{\rho}_{\boldsymbol{p}+\boldsymbol{q}} \sin\big[\ell_B^2 \frac{\boldsymbol{p}\wedge\boldsymbol{q}}{2}\big] \exp\big[\frac{\ell_B^2}{2}\boldsymbol{pq}\big],\end{aligned} \tag{5.22c}$$

based on the algebra (5.21) of the magnetic translation.

We evaluate the energy $\langle\mathfrak{S}|H|\mathfrak{S}\rangle$ by making a perturbative expansion of the spin texture around the ground state,

$$H_{\text{eff}} \equiv \langle\mathfrak{S}|H|\mathfrak{S}\rangle = \langle g_0|H|g_0\rangle - \langle g_0|[\widehat{O}, H]|g_0\rangle + \cdots . \tag{5.23}$$

The Hamiltonian H contains the physical degree of freedom associated with both the relative coordinate $\boldsymbol{R}$ and the guiding center $\boldsymbol{X}$. Since they commute each other, the relative coordinate $\boldsymbol{R}$ is easily eliminated by operating it to the ground state $|g_0\rangle$ or $\langle g_0|$. The result is

$$H_{\text{eff}} = \langle g_0|\widehat{H}|g_0\rangle - \langle g_0|[\widehat{O}, \widehat{H}]|g_0\rangle + \cdots , \tag{5.24}$$

where $\widehat{H}$ is the LLL-projected Hamiltonian. For instance, the Coulomb energy is projected as

$$\widehat{H}_C = \frac{e^2}{2\varepsilon}\int d^2x d^2y \varrho(\boldsymbol{x})\langle\!\langle V(\boldsymbol{x}-\boldsymbol{y})\rangle\!\rangle\varrho(\boldsymbol{y}) = \pi\int d^2q V(\boldsymbol{q})\widehat{\rho}_{-\boldsymbol{q}}\widehat{\rho}_{\boldsymbol{q}}, \tag{5.25}$$

where $V(\boldsymbol{q})$ is the Fourier transformation of the potential $V(\boldsymbol{x}) = 1/|\boldsymbol{x}|$. Making a straightforward algebraic calcutaion in (5.24), making a gradient expansion and taking the lowest order term in $f^a_{-\boldsymbol{q}}$, we obtain [8, 9]

$$H_{\text{eff}} = \frac{1}{2}\rho_s \sum_a \int d^2x [\partial_k s^a(\boldsymbol{x})]^2 - \frac{\rho_0}{2} g^* \mu_B B \int d^2x s^z(\boldsymbol{x}), \tag{5.26}$$

in terms of the sigma field. It is consistent with another perturbative result found in an earlier reference [19]. The first term represents the spin stiffness due to the Coulomb exchange energy, $\rho_s = \nu e^2/(16\sqrt{2\pi}\varepsilon\ell_B)$. It arises because of the following reason: The local spin rotation has components in higher Landau level since it is not a symmetry of the Hamiltonian (5.3). Only its LLL component is excited at sufficiently low temperature. Since the LLL components of the spin operators and the density operator do not commute as in (5.22b), the local spin rotation induces a local density modulation and affects the Coulomb energy.

5.4 Goldstone Mode

The Zeeman effect is quite small in actual samples. We consider the vanishing limit of the Zeeman term ($g^* = 0$). According to the effective Hamiltonian (5.26), the energy is minimized for any constant value of the sigma field, $\boldsymbol{s}(\boldsymbol{x}) = \boldsymbol{s}_0$ =constant. Hence, there exists a degeneracy in the ground states as indexed by $\boldsymbol{s}_0$. The choice of a ground state implies a spontaneous magnetization, or a *quantum Hall ferromagnetism*. When a continuous symmetry is spontaneously broken, there should arise a gapless mode known as the Goldstone mode. Quantum coherence develops spontaneously.

Actually, there is a small Zeeman effect in actual samples, and the ground state $|g_0\rangle$ is chosen where $\boldsymbol{s}_0 = (0, 0, 1)$. However, we can treat the Zeeman interaction as a perturbation because it is less important than the Coulomb interaction. The effective Hamiltonian (5.26) is reliable also in the presence of the Zeeman interaction ($g^* \neq 0$), though the Goldstone mode is no longer gapless. The key property of the QH ferromagnet is that it is a coherent state, where all spin components $S^x(\boldsymbol{x})$, $S^y(\boldsymbol{x})$ and $S^z(\boldsymbol{x})$ are simultaneously observable, $s^a(\boldsymbol{x}) = 2\rho_0^{-1}\langle S^a(\boldsymbol{x})\rangle$. An evidence is the existence of coherent excitations such as Skyrmions.

The Goldstone mode describes small fluctuations of the CP^1 field around the ground state (5.15). Up to the lowest order of the perturbation in the CP^1 field $\boldsymbol{n}(\boldsymbol{x})$, it is parametrized as [8]

$$n^{\uparrow}(\boldsymbol{x}) = 1, \qquad n^{\downarrow}(\boldsymbol{x}) = \frac{\zeta(\boldsymbol{x})}{\sqrt{\rho_0}}, \tag{5.27}$$

with $[\zeta^\dagger(\boldsymbol{x}), \zeta(\boldsymbol{y})] = i\delta(\boldsymbol{x} - \boldsymbol{y})$. The LLL condition (5.12) yields two conditions,

$$\frac{\partial}{\partial z^*}\varphi(\boldsymbol{x})|\mathfrak{S}\rangle = 0, \quad \frac{\partial}{\partial z^*}\zeta(\boldsymbol{x})|\mathfrak{S}\rangle = 0, \tag{5.28}$$

up to this order. Although they look similar, they describe very different excitation modes. As in the spinless QH system, $\varphi(\boldsymbol{x})$ is a nonlocal field and generates extended objects. On the other hand, $\zeta(\boldsymbol{x})$ is a local field, and it describes the Goldstone mode.

We may relate $\zeta(\boldsymbol{x})$ to the classical sigma field (5.10),

$$\langle\zeta(\boldsymbol{x})\rangle = \frac{\sqrt{\rho_0}}{2}\{\sigma(\boldsymbol{x}) + i\vartheta(\boldsymbol{x})\}. \tag{5.29}$$

The effective Hamiltonian (5.26) is recognized as a classical counterpart of the quantum version,

$$H_{\text{eff}} = \frac{2\rho_s}{\rho_0} \int d^2x \big[\partial_k \zeta^\dagger(\boldsymbol{x})\partial_k \zeta(\boldsymbol{x}) + \xi_L^{-2}\zeta^\dagger(\boldsymbol{x})\zeta(\boldsymbol{x})\big], \tag{5.30}$$

on the coherent state. Here, ξ_L is the coherent length,

$$\xi_L = \sqrt{\frac{2\rho_s}{g^*\mu_B B\rho_0}} = \frac{(2\pi)^{1/4}\ell_B}{2\sqrt{2\widetilde{g}}}, \tag{5.31}$$

where $\widetilde{g} = g^*\mu_B B/(e^2/\varepsilon\ell_B)$ is the ratio of the Zeeman energy to the Coulomb energy. We have $\xi_L \sim 4\ell_B$ in typical samples at $B \simeq 10$ Tesla, where $\widetilde{g} \simeq 0.02$. In the momentum space the effective Hamiltonian reads

$$\mathcal{H}_{\text{eff}}(\boldsymbol{k}) = E_{\boldsymbol{k}}\zeta^\dagger_{\boldsymbol{k}}\zeta_{\boldsymbol{k}}, \tag{5.32}$$

with $[\zeta_{\boldsymbol{k}}, \zeta^\dagger_{\boldsymbol{l}}] = \delta(\boldsymbol{k}-\boldsymbol{l})$, and the dispersion relation is

$$E_{\boldsymbol{k}} = \frac{2\rho_s}{\rho_0}\boldsymbol{k}^2 + g^*\mu_B B. \tag{5.33}$$

The Goldstone mode has acquired a gap $E_0 = g^*\mu_B B$.

6 Topological Excitations

We analyze topological (nonperturbative) excitations on the QH ferromagnet. We use the semiclassical approximation because the algebraic analysis is so difficult without making a perturbative expansion. When the N-body wave function is factorizable, the one-point function is analytic, $\langle\varphi^\alpha(\boldsymbol{x})\rangle = \omega^\alpha(z)$. From (5.8), the one-point function is parametrized as

$$e^{-\mathcal{A}(\boldsymbol{x})}e^{i\chi(\boldsymbol{x})}\sqrt{\rho_0+\varrho(\boldsymbol{x})}\,n^\alpha(\boldsymbol{x}) = \omega^\alpha(z), \tag{6.1}$$

since $|\phi(\boldsymbol{x})|^2 = \rho_0 + \varrho(\boldsymbol{x})$. Here and hereafter, all fields are classical fields. When the wave function $\omega^\alpha(z)$ is given, the electron density $\varrho(\boldsymbol{x})$ and the spin field $S^a(\boldsymbol{x})$ are determined by this equation. There are two types of excitations associated with the U(1) part and the SU(2) part of the composite-boson field. The U(1) excitation has a characteric length ℓ_B, while the SU(2) excitation has no scale provided the Zeeman term is neglected.

6.1 Vortex Excitations

The U(1) excitation is generated on the spin-polarized ground state (5.15) when $\partial_k\chi(\boldsymbol{x}) \neq 0$ and $\partial_k n^\alpha(\boldsymbol{x}) = 0$ in (6.1). We may set $\langle\varphi^\downarrow\rangle = 0$. The one-point function $\langle\varphi^\uparrow(\boldsymbol{x})\rangle$ is essentially Abelian, and the Cauchy-Riemann equation for (6.1) yields precisely the same soliton equation (4.4). The topological charge density is given by (4.5) with an analytic function $\omega(z) = \omega^\uparrow(z)$. The U(1) excitation is the vortex. The Coulomb energy of the vortex excitation is given by (4.11) with $\alpha_C \simeq 0.39$. We should remark that the vortex excitation does not induce any spin flip. There is no antivortex excitation. Instead of it an electron is placed into the spin-down state, as would increase the Coulomb energy of the same order as the vortex excitation and the Zeeman energy, as illustrated in Fig.1(a).

6.2 Skyrmion Excitations

The SU(2) excitation is generated on the spin-polarized ground state (5.15) when $\partial_k\chi(\boldsymbol{x}) = 0$ and $\partial_k n^\alpha(\boldsymbol{x}) \neq 0$ in (6.1). The CP^1 field is

$$n^\alpha(\boldsymbol{x}) = \frac{\omega^\alpha(z)}{\sqrt{|\omega^\uparrow(z)|^2 + |\omega^\downarrow(z)|^2}}, \tag{6.2}$$

yielding the wave function

$$\Psi_{\mathrm{Skyrm}}[z, z^*] = \prod_r \begin{pmatrix} \omega^\uparrow(z_r) \\ \omega^\downarrow(z_r) \end{pmatrix}_r \mathfrak{S}_{\mathrm{LN}}[\boldsymbol{x}]. \tag{6.3}$$

A simplest choice is given by

$$\begin{pmatrix} \omega^\uparrow(z) \\ \omega^\downarrow(z) \end{pmatrix} = \sqrt{\rho_0} \begin{pmatrix} z^q \\ (\kappa/2)^q \end{pmatrix}, \tag{6.4}$$

with a positive integer q, which describes a classical Skyrmion with scale κ sitting at the origin of the system [9]. It is clear in (6.4) that the Skyrmion is reduced to the vortex in the limit $\kappa \to 0$. We use this property to check the consistency of the Skyrmion theory.

The classical sigma field (5.10) is calculated as

$$s^x = \sqrt{1-(s^z)^2}\cos(q\theta), \quad s^y = -\sqrt{1-(s^z)^2}\sin(q\theta), \quad s^z = \frac{r^{2q} - (\ell_B\kappa)^{2q}}{r^{2q} + (\ell_B\kappa)^{2q}}, \tag{6.5}$$

and the Pontryagin number density (5.11) as

$$Q_0^P(\boldsymbol{x}) = \frac{q^2}{\pi}\frac{r^{2q-2}(\ell_B\kappa)^{2q}}{[r^{2q} + (\ell_B\kappa)^{2q}]^2}. \tag{6.6}$$

The spin flips at the Skyrmion center, $\boldsymbol{s} = (0,0,-1)$ at $r = 0$, while the spin-polarized ground state is approached away from it, $\boldsymbol{s} = (0,0,1)$ for $r \gg \kappa\ell_B$.

The Skyrmion excitation modulates not only the SU(2) part but also the U(1) part via the relation (6.1). The Cauchy-Rieman equation reads

$$\partial_j\big(\mathcal{A}(\boldsymbol{x}) - \ln\sqrt{\rho_0 + \varrho(\boldsymbol{x})}\big) = -\varepsilon_{jk}K_k, \tag{6.7}$$

where

$$K_k = -i\sum_\alpha n^{\alpha *}\partial_k n^\alpha. \tag{6.8}$$

From (6.7) the same soliton equation as (4.4) is derived,

$$\frac{\nu}{4\pi}\boldsymbol{\nabla}^2 \ln\Big(1 + \frac{\varrho(\boldsymbol{x})}{\rho_0}\Big) - \varrho(\boldsymbol{x}) = \nu Q_0^P(\boldsymbol{x}), \tag{6.9}$$

but the topological charge density now reads

$$Q_0^P(\boldsymbol{x}) = \frac{1}{2\pi}\varepsilon_{jk}\partial_j K_k(\boldsymbol{x}). \tag{6.10}$$

It is a straightforward calculation [18] to show that the charge (6.10) is identical to the Pontryagin number density (5.11).

The topological charge is evaluated as

$$Q^P = \int d^2x Q_0^P(\boldsymbol{x}) = \frac{1}{2\pi i}\oint dx_j K_j(\boldsymbol{x}), \tag{6.11}$$

where the loop integration $\oint$ is made to encircle the excitation at infinity ($|\boldsymbol{x}| \to \infty$). The electron number associated with the topological soliton is

$$\Delta N = \int d^2x \varrho(\boldsymbol{x}) = -\nu Q^P. \tag{6.12}$$

The topological charge is determined by the asymptotic value of the CP^1 field (6.2). We find $Q^P = q$ for the Skyrmion (6.5). The electron number of this soliton is $\Delta N = -\nu q$: It represents the number of electrons removed by the Skyrmion excitation.

The number of flipped spins is given by

$$\Delta N_s = -\int d^2x \big\{2S_z(\boldsymbol{x}) - \rho_0\big\} + \Delta N = \int d^2x \rho(\boldsymbol{x})\big\{1 - s_z(\boldsymbol{x})\big\}, \tag{6.13}$$

where ΔN is given by (6.12). We have subtracted it since it represents the number of electrons removed and not of flipped spins. The necessity of the subtraction is clear in the vortex limit $\kappa \to 0$, where it follows that $\Delta N_s = 0$ thanks to this term, as should be the case since no spin flip occurs in the vortex excitation.

The number of flipped spins (6.13) would diverge logarithmically for the Skyrmion (6.5) with $q = 1$. This is an illusion since the the Zeeman term breaks the spin SU(2) symmetry explicitly and introduces a coherent length ξ_L into the SU(2) component. The Skyrmion configuration (6.5) is valid only within the coherent domain because the coherent behavior of the spin texture is lost outside it. By cutting the upper limit of the integration at $r \simeq \xi_L$ in (6.13), we obtain

$$\Delta N_s = \kappa^2 \ln\Big(\frac{\xi_L^2}{\kappa^2 \ell_B^2} + 1\Big) - 1, \tag{6.14}$$

with the coherent length ξ_L given by (5.31).

The density modulation around the Skyrmion is governed by the soliton equation (6.9). This equation has formally the same expression as the soliton equation (4.4) for the vortex excitation. We may obtain an approximate solution in the two limits, the large Skyrmion limit ($\kappa \gg 1$) and the small Skyrmion limit ($\kappa \ll 1$). First, in the large limit we may solve (6.9) iteratively as

$$\varrho(\boldsymbol{x}) = -\nu Q_0^P(\boldsymbol{x}) - \frac{\nu^2}{8\pi\rho_0}\nabla^2 Q_0^P(\boldsymbol{x}) + \cdots. \tag{6.15}$$

We may approximate it as

$$\varrho(\boldsymbol{x}) \simeq -\nu Q_0^P(\boldsymbol{x}) = \frac{\nu}{\pi}\frac{(\ell_B\kappa)^2}{[r^2 + (\ell_B\kappa)^2]^2}, \qquad \text{for} \quad \kappa \gg 1, \tag{6.16}$$

for the Skyrmion with $q = 1$, where we have used (6.6). It agrees with the formula due to Sondhi et al. [10]. We emphasize that this formula is valid only in the large Skyrmion limit. On the other hand, the topological charge $Q_0^P(\boldsymbol{x})$ is localized in the small limit, $Q_0^P(\boldsymbol{x}) \to q\delta(\boldsymbol{x})$ as $\kappa \to 0$ in (6.6). Hence, the solution is given by the vortex configuration,

$$\varrho(\boldsymbol{x}) \simeq -\rho_0\Big(1 + \frac{\sqrt{2}r}{\ell_B} - \frac{r^2}{3\ell_B^2}\Big)e^{-\sqrt{2}r/\ell_B}, \qquad \text{for} \quad \kappa \ll 1, \tag{6.17}$$

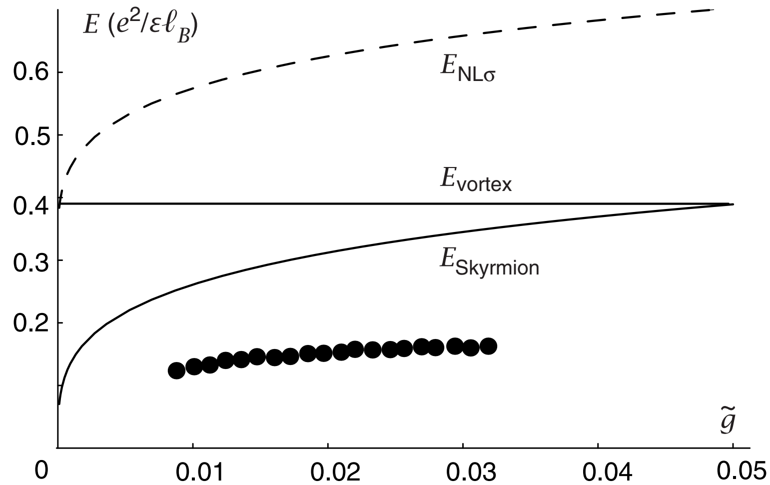

Figure 4: The activation energy is represented by the thick curve for one Skyrmion (E_{Skyrmion}) and by the thin horizontal line for one vortex (E_{vortex}) in unit of $e^2/\varepsilon\ell_B$. The dashed curve represents the Skyrmion activation energy based on the NLσ model (7.2). The filled circles are taken from the experimental data due to Schmeller et al. normalized for one quasiparticle excitation.

which has been derived in (4.13).

We evaluate the activation energy of a Skyrmion. In the semiclassical approximation it consists of the electrostatic term and the Zeeman term in (5.4),

$$E_{\mathrm{Skyrmion}} = \frac{e^2}{2\varepsilon}\int d^2x d^2y \frac{\varrho(\boldsymbol{x})\varrho(\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|} + \frac{1}{2}g^*\mu_B B\Delta N_s, \tag{6.18}$$

with (6.13). This can be calculated by using (6.16) and (6.14) for a large Skyrmion,

$$E_{\mathrm{Skyrmion}} = \frac{e^2}{\varepsilon\ell_B}\Big[\frac{\beta\nu^2}{\kappa} + \frac{\widetilde{g}}{2}\kappa^2\ln\Big(\frac{\xi_L^2}{\kappa^2\ell_B^2}+1\Big) - \frac{\widetilde{g}}{2}\Big], \tag{6.19}$$

with $\beta = 3\pi^2/64$. For a sufficiently small Skyrmion where $\Delta N_s = 0$, the Coulomb energy is calculated with the vortex configuration and is given by (4.11).

The Coulomb energy increases for a smaller Skyrmion while the Zeeman energy increases for a larger Skyrmion. The optimized scale κ is obtained so as to minimize the total energy (6.19),

$$\kappa \simeq \beta^{1/3}\big(\widetilde{g}\ln(\widetilde{g}^{-1})\big)^{-1/3}, \tag{6.20}$$

for $\widetilde{g} \ll 1$. It yields $\kappa \simeq 1.8$ for $\widetilde{g} = 0.02$. The Skyrmion scale is of the order of $1.8\ell_B$ at 10 Tesla. The number of flipped spins is calculated by (6.14), as gives $\Delta N_s \simeq 4.7$ for $\widetilde{g} = 0.02$. The Skyrmion activation energy is calculated by (6.18), which we illustrate in Fig.6.2.

A Skyrmion is a quasihole and an anti-Skyrmion is a quasielectron in the QH ferromagnet. We can make a theory of anti-Skyrmions from its microscopic wave function just as we have constructed a theory of antivortices. Though a detailed theory is yet to come, it is clear that the large scale structure is the standard one, which is obtained just by reversing the sign of s^y in (6.5),

$$s^x = \sqrt{1-(s^z)^2}\cos(q\theta), \quad s^y = \sqrt{1-(s^z)^2}\sin(q\theta), \quad s^z = \frac{r^{2q}-(\ell_B\kappa)^{2q}}{r^{2q}+(\ell_B\kappa)^{2q}}. \tag{6.21}$$

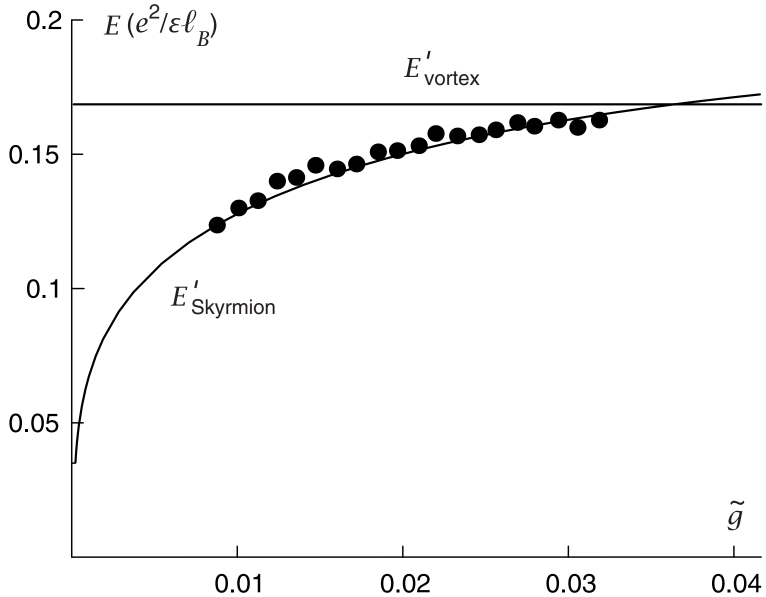

Figure 5: The best fit for the observed Skyrmion excitation energy is obtained provided the Coulomb energy is decreased by 54%, as plotted by the curve for E'_{Skyrmion} and the horizonatal line for E'_{vortex}.

The Skyrmion and anti-Skyrmion excitations are illustrated in Fig.1(b). We expect that the activation energy is nearly the same as that of the Skyrmion because they have the same electric charge except their signs. The activation energy Δ of a quasihole-quasielectron pair is observed experimentally by measuring the longitudinal resistivity (1.1). In Fig.6.2 we compare our theoretical result with the experimental data obtained by Schmeller et al. [11].

Our estimations are about two times bigger than the observed data [11]. This would be due to an oversimplification by approximating the quantum well by an infinitely thin layer. It is correct that the motion of electrons into the z axis is frozen completely at sufficiently low temperature. However, it is not justified to assume that electric charges are localized within an infinitely thin layer. The width of the layer is of the same order of the vortex-core size $\sim \ell_B$. Its effect reduces the Coulomb energy by spreading the charge into a wider domain. The best fit is obtained by replacing α_C with $0.46\alpha_C$ in (4.11), as is shown in Fig.6.2. The number of flipped spins would be $\Delta N_s \simeq 3.2$ at $\widetilde{g} = 0.02$, as is also a reasonable estimation compared with the data [11].

7 Discussion

We have presented an improved composite-boson theory of QH states, where the field operator describes solely the physical degree of freedom representing the deviation from the ground state. We have successfully analyzed excited states as well as the ground state. In this scheme the semiclassical properties of topological solitons, vortices and Skyrmions, are closely related to their microscopic wave functions.

The improved composite-boson theory is based on the bosonic field operator $\varphi(\boldsymbol{x})$ defined by (2.9). Such an operator was first considered by Read [4] to construct a Landau-Ginzburg model of QH effects. However, his effective theory is quite complicated and so different from

our microscopic theory. This operator was revived by Rajaraman et al. [17]. Although the operator itself is identical, our conclusions are significantly different from theirs. According to their conclusions, their formalism is not a unitary theory, it is useful without the LLL projection and only the ground state is successfully analyzed. On the contrary, our theory is unitary together with an integration measure, and the LLL projection has played the key role. First, it transforms the Abelian translational group into the magnetic translational group, as is the origin of the Coulomb exchange energy and leads to the NLσ model describing the Goldstone mode in the QH ferromagnet. Second, we have introduced vortices and Skyrmions merely as excitations allowed in the lowest Landau level. They induce electron density modulations according to the soliton equation which is the semiclassical LLL condition.

Skyrmions are generic SU(2) excitations in the lowest Landau level. The classical configuration is determined from its microscopic wave function. This is to be contrasted with the standard approach [10], where it is introduced as a classical solution in the NLσ model (5.26). Thus, the Skyrmion semiclassical energy is

$$E^{\text{ours}}_{\text{Skyrmion}} = E_C + E_Z \tag{7.1}$$

in our theory, where E_C and E_Z is the semiclassical Coulomb and Zeeman energies in (6.18), while it is

$$E^{\text{theirs}}_{\text{Skyrmion}} = \frac{1}{2}\rho_s \sum_a \int d^2x [\partial_k s^a(\boldsymbol{x})]^2 + E_C + E_Z \tag{7.2}$$

in the standard literature [10, 9]. We make comments on this point. As we have stated, there are two complementary methods to estimate the energy $\langle \mathfrak{S}|H|\mathfrak{S}\rangle$ of the spin texture. The algebraic method is appropriate for the analysis of small perturbative fluctuations around the ground state, where the NLσ-model term has been derived [8, 9] in the lowest order approximation. On the other hand, the semiclassical approximation is a powerful method for the analysis of nonperturbative excitations, where there is no reason to include the NLσ-model term. Indeed, its absence is required from the consistency condition that the Skyrmion wave function is reduced to the vortex wave function in the limit $\kappa \to 0$. We have checked that, as $\kappa \to 0$, the Skyrmion excitation energy is reduced precisely to the vortex excitation energy based on our formula (7.1). We also point out that, accordind to their formula (7.2), due to the NLσ-model term the Skyrmion excitation energy would become larger than the vortex excitation energy for almost all values of $\widetilde{g}$, as is seen in Fig.6.2. This means that Skyrmion would not be relevant excitations, as contradicts experimental observations of Skyrmions.

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